Exercise: ELBO for GMMs

(Source: Exercise 10.16 of (**?**).).

Consider VBEM for GMMs. Show that the lower bound has the following form

$$\mathcal{L} = \mathbb{E}\left[\ln p(\mathbf{x}|\mathbf{z},\boldsymbol{\mu},\boldsymbol{\Lambda})\right] + \mathbb{E}\left[\ln p(\mathbf{z}|\boldsymbol{\pi})\right] + \mathbb{E}\left[\ln p(\boldsymbol{\pi})\right] + \mathbb{E}\left[\ln p(\boldsymbol{\mu},\boldsymbol{\Lambda})\right]$$
$$- \mathbb{E}\left[\ln q(\mathbf{z})\right] - \mathbb{E}\left[\ln q(\boldsymbol{\pi})\right] - \mathbb{E}\left[\ln q(\boldsymbol{\mu},\boldsymbol{\Lambda})\right] \tag{1}$$

where

$$\mathbb{E}\left[\ln p(\mathbf{x}|\mathbf{z},\boldsymbol{\mu},\boldsymbol{\Lambda})\right] = {}^1\!/_2 \sum_k N_k \Big\{ \ln\tilde{\Lambda}_k - D\tilde{\kappa}_k^{-1} - \nu_k \mathrm{tr}(\mathbf{S}_k\mathbf{L}_k)$$
$$-\nu_k(\overline{\mathbf{x}}_k - \mathbf{m}_k)^T\mathbf{L}_k(\overline{\mathbf{x}}_k - \mathbf{m}_k) - D\ln(2\pi)\Big\} \tag{2}$$

$$\mathbb{E}\left[\ln p(\mathbf{z}|\boldsymbol{\pi})\right] = \sum_n\sum_k r_{nk}\ln\tilde{\pi}_k \tag{3}$$

$$\mathbb{E}\left[\ln p(\boldsymbol{\pi})\right] = \ln C_{dir}(\boldsymbol{\alpha}_0) + (\alpha_0 - 1)\sum_k \ln\tilde{\pi}_k \tag{4}$$

$$\mathbb{E}\left[\ln p(\boldsymbol{\mu},\boldsymbol{\Lambda})\right] = {}^1\!/_2 \sum_k \Big\{ D\ln(\kappa/2\pi) + \ln\tilde{\Lambda}_k - \frac{D\kappa}{\tilde{\kappa}_k}$$
$$-\kappa\nu_k(\mathbf{m}_k - \mathbf{m}_0)^T\mathbf{L}_k(\mathbf{m}_k - \mathbf{m}_0)$$
$$+\ln C_{Wi}(\mathbf{L}_0,\nu_0) + \frac{\nu_0 - D - 1}{2}\ln\tilde{\Lambda}_k - {}^1\!/_2\nu_k\mathrm{tr}(\mathbf{L}_0^{-1}\mathbf{L}_k)\Big\} \tag{5}$$

$$\mathbb{E}\left[\ln q(\mathbf{z})\right] = \sum_n\sum_k r_{nk}\ln r_{nk} \tag{6}$$

$$\mathbb{E}\left[\ln q(\boldsymbol{\pi})\right] = \sum_k(\alpha_k - 1)\ln\tilde{\pi}_k + \ln C_{dir}(\boldsymbol{\alpha}) \tag{7}$$

$$\mathbb{E}\left[\ln q(\boldsymbol{\mu},\boldsymbol{\Lambda})\right] = \sum_k\left\{ {}^1\!/_2\ln\tilde{\Lambda}_k + \frac{D}{2}\ln\left(\frac{\tilde{\kappa}_k}{2\pi}\right) - \frac{D}{2} - \mathbb{H}\left(q(\boldsymbol{\Lambda}_k)\right)\right\} \tag{8}$$

where

$$\ln\tilde{\Lambda}_k \triangleq \mathbb{E}\left[\ln|\boldsymbol{\Lambda}_k|\right] \tag{9}$$
$$\ln\tilde{\pi}_k \triangleq \mathbb{E}\left[\ln\pi_k\right] \tag{10}$$