## Exercise: GSM representation of group lasso

Consider the prior $\tau_j^2 \sim \text{Ga}(\delta, \rho^2/2)$, ignoring the grouping issue for now. The marginal distribution induced on the weights by a Gamma mixing distribution is called the **normal Gamma** distribution and is given by

$$\text{NG}(w_j|\delta,\rho) = \int \mathcal{N}(w_j|0,\tau_j^2)\text{Ga}(\tau_j^2|\delta,\rho^2/2)d\tau_j^2 \quad (1)$$

$$= \frac{1}{Z}|w_j|^{\delta-1/2}\,\mathcal{K}_{\delta-1/2}(\rho|w_j|) \quad (2)$$

$$1/Z = \frac{\rho^{\delta+1/2}}{\sqrt{\pi}\,2^{\delta-1/2}\,\rho(\delta)} \quad (3)$$

where $\mathcal{K}_\alpha(x)$ is the modified Bessel function of the second kind (the besselk.m function in Matlab).

Now suppose we have the following prior on the variances

$$p(\boldsymbol{\sigma}_{1:D}^2) = \prod_{g=1}^{G} p(\boldsymbol{\sigma}_{1:d_g}^2),\ \ p(\boldsymbol{\sigma}_{1:d_g}^2) = \prod_{j\in g}\text{Ga}(\tau_j^2|\delta_g,\rho^2/2) \quad (4)$$

The corresponding marginal for each group of weights has the form

$$p(\mathbf{w}_g) \propto |u_g|^{\delta_g - d_g/2}\,\mathcal{K}_{\delta_g-d_g/2}(\rho u_g) \quad (5)$$

where

$$u_g \triangleq \sqrt{\sum_{j\in g} w_{g,j}^2} = ||\mathbf{w}_g||_2 \quad (6)$$

Now suppose $\delta_g = (d_g+1)/2$, so $\delta_g - d_g/2 = 1/2$. Conveniently, we have $\mathcal{K}_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\exp(-z)$. Show that the resulting MAP estimate is equivalent to group lasso.