Exercise: Mutual information for correlated normals

(Source: (**?**, Q9.3).)

Find the mutual information $I(X_1, X_2)$ where $\mathbf{X}$ has a bivariate normal distribution:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\right) \tag{1}$$

Evaluate $I(X_1, X_2)$ for $\rho = 1$, $\rho = 0$ and $\rho = -1$ and comment. Hint: The (differential) entropy of a d-dimensional Gaussian is

$$h(\mathbf{X}) = {}^{1}\!/_{2} \log_2 \left[(2\pi e)^d \det \Sigma\right] \tag{2}$$

In the 1d case, this becomes

$$h(X) = {}^{1}\!/_{2} \log_2 \left[2\pi e \sigma^2\right] \tag{3}$$

Hint: $\log(0) = \infty$.