Exercise: EM for ARD

Derive the EM algorithm for fitting a linear regression model with an ARD prior. Compute the type II MLE as well as the type II MAP estimate using the priors $\alpha_j \sim \mathrm{Ga}(a, b)$ and $\beta \sim \mathrm{Ga}(c, d)$.

Hint: the following identity should be useful

$$\begin{aligned}
\boldsymbol{\Sigma}\mathbf{X}^T\mathbf{X} &= \boldsymbol{\Sigma}\mathbf{X}^T\mathbf{X} + \beta^{-1}\boldsymbol{\Sigma}\mathbf{A} - \beta^{-1}\boldsymbol{\Sigma}\mathbf{A} && (1)\\
&= \boldsymbol{\Sigma}(\mathbf{X}^T\mathbf{X}\beta + \mathbf{A})\beta^{-1} - \beta^{-1}\boldsymbol{\Sigma}\mathbf{A} && (2)\\
&= (\mathbf{A} + \beta\mathbf{X}^T\mathbf{X})^{-1}(\mathbf{X}^T\mathbf{X}\beta + \mathbf{A})\beta^{-1} - \beta^{-1}\boldsymbol{\Sigma}\mathbf{A} && (3)\\
&= (\mathbf{I} - \mathbf{A}\boldsymbol{\Sigma})\beta^{-1} && (4)
\end{aligned}$$