

Exercise: Convolution of two Gaussians is a Gaussian

Show that the convolution of two Gaussians is a Gaussian, i.e.,

$$p(y) = \mathcal{N}(x_1|\mu_1, \sigma_1^2) \otimes \mathcal{N}(x_2|\mu_2, \sigma_2^2) = \mathcal{N}(y|\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \quad (1)$$

where $y = x_1 + x_2$, $x_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $x_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.