Exercise: Gaussian DAGs vs Gaussian MRFs

(Source: MacKay.)

Consider a Gaussian DAG in which each CPD has the form

$$X_j = \mu_j + \sum_{k \in \pi_j} w_{jk}(X_k - \mu_k) + \sqrt{v_j} Z_j \tag{1}$$

Let $\mu_j = 0$ and $v_j = 1$ for all j. Define the matrix

$$\mathbf{T} = \begin{pmatrix} 1 & & & & \\ -w_{21} & 1 & & & \\ -w_{32} & -w_{31} & 1 & & \\ \vdots & & & \ddots & \\ -w_{d1} & -w_{d2} & \dots & -w_{d,d-1} & 1 \end{pmatrix}$$

Let $\mathbf{M} = \mathbf{T}^\top$ be upper triangular. Then we have $\mathbf{\Omega} = \mathbf{\Sigma}^{-1} = \mathbf{M}\mathbf{M}^\top$ (since $\mathbf{D} = \mathrm{diag}(v_j) = \mathbf{I}$). Note that $\mathbf{M}_{:,j}$ are the weights into node j, and $\mathbf{M}_{i,:}$ are the weights out of node i.

Prove or disprove the following statement:

$$M_{i,j} = 0 \quad \forall j > i \implies \Omega_{i,j} = 0 \tag{2}$$