Exercise: Normalization constant for a 1D Gaussian

The normalization constant for a zero-mean Gaussian is given by

$$Z = \int_a^b \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \tag{1}$$

where $a = -\infty$ and $b = \infty$. To compute this, consider its square

$$Z^2 = \int_a^b \int_a^b \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) dxdy \tag{2}$$

Let us change variables from cartesian (x, y) to polar (r, θ) using $x = r\cos\theta$ and $y = r\sin\theta$. Since $dxdy = rdrd\theta$, and $cos^2\theta + \sin^2\theta = 1$, we have

$$Z^2 = \int_0^{2\pi} \int_0^{\infty} r\exp\left(-\frac{r^2}{2\sigma^2}\right) drd\theta \tag{3}$$

Evaluate this integral and hence show $Z = \sqrt{\sigma^2 2\pi}$. Hint 1: separate the integral into a product of two terms, the first of which (involving $d\theta$) is constant, so is easy. Hint 2: if $u = e^{-r^2/2\sigma^2}$ then $du/dr = -\frac{1}{\sigma^2} re^{-r^2/2\sigma^2}$, so the second integral is also easy (since $\int u'(r)dr = u(r)$).