Exercise: Constant factor optimality for alpha-expansion

Let $\mathcal{X}$ be a pairwise metric Markov random field over a graph $G = (V, E)$. Suppose that the variables are nonbinary and that the node potentials are nonnegative. Let $\mathcal{A}$ denote the set of labels for each $X \in \mathcal{X}$. Though it is not possible to (tractably) find the globally optimal assignment $x^\star$ in general, the α-expansion algorithm provides a method for finding assignments $\hat{x}$ that are locally optimal with respect to a large set of transformations, *i.e.*, the possible α-expansion moves.

Despite the fact that α-expansion only produces a locally optimal MAP assignment, it is possible to prove that the energy of this assignment is within a known factor of the energy of the globally optimal solution $x^\star$. In fact, this is a special case of a more general principle that applies to a wide variety of algorithms, including max-product belief propagation and more general move-making algorithms: If one can prove that the solutions obtained by the algorithm are 'strong local minima', *i.e.*, local minima with respect to a large set of potential moves, then it is possible to derive bounds on the (global) suboptimality of these solutions, and the quality of the bounds will depend on the nature of the moves considered. (There is a precise definition of 'large set of moves'.)

Consider the following approach to proving the suboptimality bound for α-expansion (details are in (**?**) if you get stuck).

1. Let $\hat{x}$ be a local minimum with respect to expansion moves. For each $\alpha \in \mathcal{A}$, let $V^\alpha = \{s \in V \mid x_s^\star = \alpha\}$, *i.e.*, the set of nodes labelled α in the global minimum. Let x' be an assignment that is equal to $x^\star$ on V^α and equal to $\hat{x}$ elsewhere; this is an α-expansion of $\hat{x}$. Verify that $E(x^\star) \leq E(\hat{x}) \leq E(x')$.

2. Building on the previous part, show that $E(\hat{x}) \leq 2cE(x^\star)$, where $c = \max_{(s,t) \in E} \left(\frac{\max_{\alpha \neq \beta} \varepsilon_{st}(\alpha, \beta)}{\min_{\alpha \neq \beta} \varepsilon_{st}(\alpha, \beta)} \right)$ and E denotes the energy of an assignment.

   Hint. Think about where x' agrees with $\hat{x}$ and where it agrees with $x^\star$.