Exercise: Deriving the residual error for PCA

1. Prove that

$$||\mathbf{x}_i - \sum_{j=1}^{K} z_{ij}\mathbf{v}_j||^2 = \mathbf{x}_i^T\mathbf{x}_i - \sum_{j=1}^{K} \mathbf{v}_j^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{v}_j \tag{1}$$

   Hint: first consider the case $K = 2$. Use the fact that $\mathbf{v}_j^T\mathbf{v}_j = 1$ and $\mathbf{v}_j^T\mathbf{v}_k = 0$ for $k \neq j$. Also, recall $z_{ij} = \mathbf{x}_i^T\mathbf{v}_j$.

2. Now show that

$$J_K \triangleq \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^T\mathbf{x}_i - \sum_{j=1}^{K}\mathbf{v}_j^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^T\mathbf{x}_i - \sum_{j=1}^{K}\lambda_j \tag{2}$$

   Hint: recall $\mathbf{v}_j^T\mathbf{C}\mathbf{v}_j = \lambda_j\mathbf{v}_j^T\mathbf{v}_j = \lambda_j$.

3. If $K = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $K < d$ terms is given by

$$J_K = \sum_{j=K+1}^{d}\lambda_j \tag{3}$$

   Hint: partition the sum $\sum_{j=1}^{d}\lambda_j$ into $\sum_{j=1}^{K}\lambda_j$ and $\sum_{j=K+1}^{d}\lambda_j$.