

### Exercise: Deriving the second principal component

1. Let

$$J(\mathbf{v}_2, \mathbf{z}_2) = \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i - z_{i1}\mathbf{v}_1 - z_{i2}\mathbf{v}_2)^T (\mathbf{x}_i - z_{i1}\mathbf{v}_1 - z_{i2}\mathbf{v}_2) \quad (1)$$

Show that  $\frac{\partial J}{\partial \mathbf{z}_2} = 0$  yields  $z_{i2} = \mathbf{v}_2^T \mathbf{x}_i$ .

2. Show that the value of  $\mathbf{v}_2$  that minimizes

$$\tilde{J}(\mathbf{v}_2) = -\mathbf{v}_2^T \mathbf{C} \mathbf{v}_2 + \lambda_2 (\mathbf{v}_2^T \mathbf{v}_2 - 1) + \lambda_{12} (\mathbf{v}_2^T \mathbf{v}_1 - 0) \quad (2)$$

is given by the eigenvector of  $\mathbf{C}$  with the second largest eigenvalue. Hint: recall that  $\mathbf{C} \mathbf{v}_1 = \lambda_1 \mathbf{v}_1$  and  $\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^T) \mathbf{x}$ .