Exercise: PPCA variance terms

Recall that in the PPCA model, $\mathbf{C} = \mathbf{W}\mathbf{W}^T + \sigma^2\mathbf{I}$. We will show that this model correctly captures the variance of the data along the principal axes, and approximates the variance in all the remaining directions with a single average value σ^2.

Consider the variance of the predictive distribution $p(\mathbf{x})$ along some direction specified by the unit vector $\mathbf{v}$, where $\mathbf{v}^T\mathbf{v} = 1$, which is given by $\mathbf{v}^T C\mathbf{v}$.

1. First suppose $\mathbf{v}$ is orthogonal to the principal subspace. and hence $\mathbf{v}^T\mathbf{U} = \mathbf{0}$. Show that $\mathbf{v}^T\mathbf{C}\mathbf{v} = \sigma^2$.

2. Now suppose $\mathbf{v}$ is parallel to the principal subspace. and hence $\mathbf{v} = \mathbf{u}_i$ for some eigenvector $\mathbf{u}_i$. Show that $\mathbf{v}^T\mathbf{C}\mathbf{v} = (\lambda_i - \sigma^2) + \sigma^2 = \lambda_i$.