Exercise: ELBO for univariate Gaussians

Considering computing the posterior for the parameters of a univariate Gaussian, $p(\mu, \lambda|\mathcal{D})$, using the prior

$$p(\mu, \lambda) = \mathcal{N}(\mu|\mu_0, (\kappa_0\lambda)^{-1})\mathrm{Ga}(\lambda|a_0, b_0) \tag{1}$$

and a fully factored posterior

$$q(\mu, \lambda) = \mathcal{N}(\mu|\mu_N, \kappa_N^{-1})\gamma(\lambda|a_N, b_N) \tag{2}$$

Show that the ELBO is given by

$$L(q) = \text{const} + {}^1\!/\!_2 \ln \frac{1}{\kappa_N} + \ln \Gamma(a_N) - a_N \ln b_N \tag{3}$$

Hint: the entropies of the variational posteriors are given by

$$\mathbb{H}\left(\mathcal{N}(\mu_N, \kappa_N^{-1})\right) = -{}^1\!/\!_2 \log \kappa_N + {}^1\!/\!_2(1 + \log(2\pi)) \tag{4}$$

$$\mathbb{H}\left(\mathrm{Ga}(a_N, b_N)\right) = \log\Gamma(a_N) - (a_N - 1)\psi(a_N) - \log(b_N) + a_N \tag{5}$$

where $\psi()$ is the digamma function. Also, the expectations of various terms wrt these distributions are given below.

$$\mathbb{E}\left[\log x | x \sim \mathrm{Ga}(a, b)\right] = \psi(a) - \log(b) \tag{6}$$

$$\mathbb{E}\left[x | x \sim \mathrm{Ga}(a, b)\right] = \frac{a}{b} \tag{7}$$

$$\mathbb{E}\left[x | x \sim \mathcal{N}(\mu, \sigma^2)\right] = \mu \tag{8}$$

$$\mathbb{E}\left[x^2 | x \sim \mathcal{N}(\mu, \sigma^2)\right] = \mu^2 + \sigma^2 \tag{9}$$