Exercise: Variance and MSE of estimators for Gaussian variance

Prove that the standard error for the MLE for a Gaussian variance is

$$\sqrt{\mathbb{V}\left[\hat{\sigma}^2_{\text{mle}}\right]} = \sqrt{\frac{2(N-1)}{N^2}}\sigma^2 \tag{1}$$

Hint: use the fact that

$$\frac{N-1}{\sigma^2}\hat{\sigma}^2_{\text{unb}} \sim \chi^2_{N-1}, \tag{2}$$

and that $\mathbb{V}\left[\chi^2_{N-1}\right] = 2(N-1)$. Finally, show that $\text{MSE}(\hat{\sigma}^2_{\text{unb}}) = \frac{2N-1}{N^2}\sigma^4$ and $\text{MSE}(\hat{\sigma}^2_{\text{mle}}) = \frac{2}{N-1}\sigma^4$.